SCIENCE

CONTENTS

CONCERNING THE FIGURE AND THE DIMENSIONS OF THE UNIVERSE OF SPACE [1]

THERE is something a little incongruous in attempting to consider the subject of this address in a theater or lecture hall whose roof and walls shut out from view the wide expanses of the world and the azure deeps. For how can we, amid the familiar finite scenes of a closed and blinded room, command a fitting mood for contemplating the infinite scenes without and beyond? A subject that has sheer vastness for its central or major theme demands for its appropriate contemplation the still expanse of some vast and open solitude, such as the peak of a lone and lofty mountain would afford, where the gaze meets no wall save the far horizon and no roof but the starry sky. Perhaps you will be good enough for the time to transport yourselves, in imagination, into the stillness of such a solitude, so that in the musing spirit of the place the questions to be propounded for consideration here may arise naturally and give us a due sense of their significance and impressiveness. What are the dimensions and what is the figure of our universe of space? How big is it and what is its shape? What is the figure of it and what is its size?

I do not mind owning that these questions have haunted me a good deal from the days of my youth. It happened in

[1] An address delivered under the auspices of the local chapters of the Society of Sigma Xi at the state universities of Minnesota, Nebraska and Iowa, April 24, 28 and 30, respectively, and at a joint meeting of the chapters of Sigma Xi and Phi Beta Kappa of Columbia University, May 8.

those days, though I was not aware of it nor became aware of it till after many years, that there were then coming into mathematics, just entering the fringe, so to speak, or the vestibule of the science, certain striking ideas which, as I venture to hope we may see, were destined, if not indeed to enable us to answer the questions with certainty, at all events to clarify them, to enrich their meaning and to make it possible to discuss them profitably. It has not been my fortune to meet many persons who had seriously propounded the questions to themselves or who seemed to be immediately interested in them when propounded by others—not many, even among astronomers, whose minds, it may be assumed, are specially "accustomed to contemplation of the vast." And so I have been forced to the somewhat embarrassing conclusion that my own long interest in the questions has been due to the fact of my being of a specially practical turn of mind. Quite seriously I venture to say that we are here engaged in a practical enterprise. For even if the questions were in the nature of the case unanswerable, which we do not admit, who does not know how great the boons that have come to men through pursuit of the unattainable? And who does not know that, as Mr. Chesterton has said, if you wish really to know a man, the most practical question to ask is, not about his occupation or his club membership or his party or church affiliations, but what are his views of the all-embracing world? What does he think of the universe? Do but fancy for a moment that in somewise men should come to *know* the exact shape or figure and especially the exact size or dimensions of the all-immersing space of our universe. It requires but little imagination, not much reflection, no extensive knowledge of cosmogonic history and spec-

ulation, no very profound insight into the ways of truth to men; it needs, I say, but little philosophic sense to see that such knowledge would in a thousand ways, direct and indirect, react powerfully upon our whole intelligence, upon all our attitudes, sentiments and views, transforming our theology, our ethics, our art, our religion, our philosophy, our literature, our science, and therewith affect profoundly the whole sense and manner, the tone, color and meaning, of all our institutions and the affairs of daily life. Nothing is quite so practical, in the sense of being effectual and influential, as the views men hold, consciously or unconsciously, regarding the great locus of their lives and their cosmic home.

In order to discuss the questions before us intelligibly and profitably it is not necessary by way of clearing the ground to enter far into metaphysical speculation or into psychological analysis with a view to ascertaining what it is that we mean or ought to mean by space. We are not obliged to dispute, much less decide, whether space is subjective or objective or both or indeed something that, as Plato in the "Timæus" acutely contends, is neither the one nor the other. We may or may not agree with the contention of Kant that space is, not an object, but the form, of outer sense; we may or may not agree with the radically different contention of Poincaré that (geometric as distinguished from sensible) space is nothing but what is known in mathematics as a group, of which the concept "is imposed on us, not as form of our sense, but as form of our understanding." It is, I say, not necessary for us, in the interest of soundness and intelligibility, to try to compose such differences or to attempt a settlement of these profound and important questions. As to the distinction between

sensible space and geometric space, it would indeed be indispensable to draw it sharply and to keep it always in mind, if we were undertaking to ascertain what the subject (or the object) of geometry is, or, what is tantamount, if we were seeking to get clearly aware of what it is that geometry is about. But in discussing the subject before us it is unnecessary to be always guarding that distinction; for, whilst it is the space of geometry, and not sensible space, that we shall be talking about, yet it would be a hindrance rather than a help if we did not allow, as we habitually do allow, the two varieties of space—the imagery of the one, the conceptual characters of the other—to mingle freely in our thinking. There will be finesse enough for the keenest arrows of our thought without our going out of the way to find it. A procedure less sophisticated will suffice. It will be sufficient to regard space as being what, to the layman and to the student of natural science, it has always seemed to be: a vast region or room round about us, an immense exteriority, locus of all suspended and floating objects of outer sense, the whence, where and whither of motion, theater, in a word, of the ageless drama of the physical universe. In naturally so construing the term we do not commit ourselves to the philosophy, so-called, of common sense; we thus merely save our discourse from the encumbrance of needless refinements; for it is obvious that, if space be not indeed what we have said it seems to be, the seeming is yet a fact, and our questions would remain without essential change: what, then, we should ask, are the dimensions and what is the figure of that seeming?

Though all the things contained within that triply extended spread or expanse which we call space are subject to the law of ceaseless change, the expanse itself, the container of all, appears to suffer no variation whatever, but to be, unlike time, a genuine constant, the same yesterday, today and forever, sole absolute invariant under the infinite host of transformations that constitute the cosmic flux. Whether it be so in fact, of course we do not know. We only know that no good reason has ever been advanced for holding the contrary as an hypothesis.

And yet there is a sense, which we ought I think to notice, an interesting sense, in which space seems to be, not a constant, but, like time, a variable. There is a sense, deeper and juster perhaps than at first we suspect, in which the space of our universe has in the course of time alternately shrunken and grown. During the last century, for example, it has, so it seems, greatly grown, in response, it may be, to an increasing need of the human mind. By grown I do not mean grown in the usual sense, I do not mean the biological sense, I do not mean the sense that was present to the mind of that great man, Leonardo da Vinci, when he wrote in effect as follows: if you wish to know that the earth has been growing, you have only to observe "how, among the high mountains, the walls of ancient and ruined cities are being covered over and concealed by the earth's increase"; and, if you would learn how *fast* the earth is growing, you have only to set a vase, filled with pure earth, upon a roof; to note how green herbs will immediately begin to shoot up; to note that these, when mature, will cast their seeds; to allow the process to continue through repetition; then, after the lapse of a decade, to measure the soil's increase; and, finally, to multiply, in order to have thus determined "how much the earth has grown in the course of a thousand years." In this matter, Leo-

nardo was doubtless wrong. At all events current scientific views are against him. The earth, we know, has grown, but the growth has been by accretion, by addition from without, and not, in biologic sense, by expansion from within (unless, indeed, we adopt the beautiful hypothesis of the poet and physicist, Theodor Fechner, for which so hard-headed a scientific man as Bernhardt Riemann had so much respect, the hypothesis, namely, that the plants, the earth and the stars have souls). The myriad-minded Florentine was, we of to-day think, in error, his error being one of those brilliant mistakes that but few men have been qualified to make. But in saying that space has grown we do not mean that it has grown in the biologic sense of Leonardo nor yet in the sense of addition from without. We mean that it has grown as a thing in mind may grow, as a thing in thought may grow; we mean that it has grown in men's conception of it. That space has, in this sense, been enlarged prodigiously in the course of recent time is evident to all. It has been often said that the first grand discovery of modern times is the immense extension of the universe *in space.*" It would be juster to say that the first grand achievement of modern science has been the immense extension of space itself, the prodigious enlargement of it, in the imagination and especially in the thought of men. If we will but take the trouble to recall vividly the Mosaic cosmogony, in terms of which most of us have but recently ceased to frame our sublimest conceptions of the vast; if we remind ourselves of Plato's "concentric crystal spheres, the adamantine axis turning in the lap of necessity, the bands that held the heaven together like a girth that clasps a ship, the shaft which led from earth to sky, and which was paced by the soul in a

thousand years"; if we compare these conceptions with our own; if we think of "the fields from which our stars fling us their light," fields that are really near and yet are so far that the swiftest of messengers, capable of circling the earth eight times in a second, requires for its journey hither thousands of years; if we do but make some such comparisons, we shall begin to realize dimly that, compared with modern space— the space of modern thought—elder space —the space of elder thought—is indeed "but as a cabinet of brilliants, or rather a little jewelled cup found in the ocean or the wilderness."

Suppose that in fact space were thus, like time, not a constant, but a variable; suppose it were a mental thing growing with the growth of mind; an increasing function of increasing thought; suppose it were a thing whose enlargement is essential as a psychic condition or concomitant or effect of the progress of science; would not our questions regarding its figure and its dimensions then lose their meaning? The answer is, no; as rational beings we should still be bound to ask: what are the dimensions and what is the figure of space to date? That is not all. If these questions were answered, we could propound the further questions: whether the space so characterized—the space of the present— is adequate to the present needs of science, and whether it is not destined to yet further expansion in response to the future needs of thought.

Men do not feel, however, that such spatial enlargements as I have indicated are genuine enlargements of space. In spite of whatever metaphysics or psychology may seem obliged to say to the contrary, men feel that what is *new* in such an enlargement is merely an increase of enlightenment regarding something old; they

feel that what is new is, not an added vast-
ness, but a discovery of a vastness that al-
ways was and always will be. Let us trust
this feeling and, regarding space as con-
stant from everlasting to everlasting, let us
take the questions in their natural intent
and form: what are the dimensions and
what is the figure of our universe of space?

If you propound these questions to a
normal student of natural science, say to a
normal astronomer, his response will be—
what? If you appear to him to be quite
sincere and if, besides, he be in an amiable
mood, his response will, not improbably, be
a significant shrug of the shoulders, de-
signed to intimate that his time is too
precious to be squandered in considering
questions that, if not meaningless, are at
all events unanswerable. I maintain, on
the contrary, that this same student of nat-
ural science and, indeed, all other normally
educated men and women, have, as a part
of their intellectual stock in trade, per-
fectly definite answers to both of the ques-
tions. I do not mean that they are aware
of possessing such wealth nor shall I under-
take to say in advance whether their an-
swers be correct. What I am asserting and
what, with your assistance, I shall endeavor
to demonstrate, is that perfectly precise,
very intelligent and perfectly intelligible
answers to both of the questions are log-
ically involved in what every normally edu-
cated mind regards as the securest of its in-
tellectual possessions. In order to show
that such answers are to be found embedded
in the content of the normally educated
mind and in order to lay them bare, it will
be necessary to have recourse to the process
of explication. Explication, however, is
nothing strange to an academic audience.
It is true, indeed, that we no longer derive
the verb, to educate, from *educere,* but it is
yet a fact, as every one knows, that a large
part of education is *eduction*—the leading

forth into light what is hidden in the fa-
miliar content of our minds.

What are those answers? I shall present
them in the familiar and brilliant words of
one who in the span of a short life achieved
a seven-fold immortality: immortality as a
physicist, as a philosopher, as a mathema-
tician, as a theologian, as a writer of prose,
as an inventor and as a fanatic. From
this brief but "immortal" characteriza-
tion I have no doubt that you detect the au-
thor at once and at once recall the words:
*Space is an infinite sphere whose center is
everywhere and whose surface is nowhere.*

You will observe that, without change of
meaning, I have substituted "space" for
"universe" and "surface" for "circum-
ference." This brilliant *mot* of Blaise
Pascal, as every one knows, has long been
valued throughout the world as a splendid
literary gem. I am not aware that it has
been at any time regarded seriously as a
scientific thesis. It may, however, be so
regarded. I propose to show, with your
cooperation, that this exquisite saying of
Pascal expresses with mathematical pre-
cision the firm, albeit unconscious, con-
viction of the normally educated mind re-
specting the size and the shape of the space
of our universe. Be good enough to note
carefully at the outset the cardinal phrases:
*infinite sphere, center everywhere, surface
nowhere.*

If you are told that there is an object
completely enclosed and that the object is
equally distant from all parts of the en-
closing boundary or wall, you instantly and
rightly think of a sphere having that object
as center. Let me ask you to think of some
point, any convenient point, *P,* together
with all the straight lines or rays—called a
sheaf of lines or rays—that, beginning at
P, run out from it as far as ever the nature
of space allows. We ask: do all the rays of
the sheaf run out equally far? It seems

perfectly evident that they do, and with this we might be content. It will be worth while, however, to examine the matter a little more attentively. Denote by L any chosen line or ray of the sheaf. Choose any convenient unit of length, say a mile. We now ask: how many of our units, how many miles can we, starting from P, lay off along L? Lay off, I mean, not in fact, but in thought. In other words: how many steps, each a mile long, can we, in traversing L, take in thought? Hereafter let the phrase ''in thought'' be understood. Can the question be answered? It can. Can it be answered definitely? Absolutely so. How? As follows. Before proceeding, however, let me beg of you not to hesitate or shy if certain familiar ideas seem to get submitted to the logical process—the mind-expanding process—of generalization. There is to be no resort to any kind of legerdemain. Let us be willing to transcend imagination, and, without faltering, to follow thought, for thought, free as the spirit of creation, owns no bar save that of inconsistence or self-contradiction. Consider the sequence of cardinal numbers,

$$(S) \quad 1, 2, 3, 4, 5, 6, 7, \cdots.$$

The sequence is neither so dry nor so harmless as it seems. It has a beginning; but it has no end, for, by the law of its formation, after each term there is a next. The difference between a sequence that stops somewhere and one that has no end is awful. No one, unless spiritually unborn or dead, can contemplate that gulf without emotions that take hold of the infinite and everlasting. Let us compare the sequence with the ray L of our sheaf. Choose in (S) any number n, however large. Can we go from P along L that number n of miles? We are certain that we can. Suppose the trip made, a mile post set up and on it painted the number n to tell how far the post is from P. As n is any number in (S), we may as well suppose, indeed we have already implicitly supposed, mile posts, duly distributed and marked, to have been set up along L to match each and every number in the sequence. Have we thus set up all the mile posts that L allows? We are certain that we have, for, if we go out from P along L any possible but definite number of miles, we are perfectly certain that that number is a number in the sequence, and that accordingly the journey did but take us to a post set up before. What is the upshot? It is that L admits of precisely as many mile posts as there are cardinal numbers, neither more nor less. How long is L? The answer is: L is exactly as many miles long as there are integers or terms in the sequence (S). Can we say of any other line or ray L' of the sheaf what we have said of L? We are certain that we can. Indeed we have said it, for L was *any* line of the sheaf. May we, then, say that any two lines, L and L', of the sheaf are *equal?* We may and we must. For, just as we have established a one-to-one correspondence between the mile posts of L and the terms of (S), so we may establish a one-to-one correspondence between the mile posts of L and those of L', and what we mean by the *equality* of two classes of things is precisely the possibility of thus setting up a one-to-one correlation between them. Accordingly, all the lines or rays of our sheaf are equal. We can not fail to note that thus there is forming in our minds the conception of a sphere, centered at P, larger, however, than any sphere of slate or wood or marble—a sphere, if it be a sphere, whose radii are the rays of our sheaf. Is not the thing, however, too vast to be a sphere? Obviously yes, if the lines or rays of the sheaf have a length that is indefinite, unassignable; obviously no, if their length be assignable and definite. We have

seen the length of a ray contains exactly as many miles as there are integers or terms in (S). The question, then, is: has the totality of these terms a definite assignable number? The answer is, yes. To show it, look sharply at the following fact, a bit difficult to see only because it is so obvious, being writ, so to speak, on the very surface of the eye. I wish, in a word, to make clear what is meant by the cardinal number of any given class of things. The fingers of my right hand constitute a class of objects; the fingers of my left hand, another class. We can set up a one-to-one correspondence between the classes, pairing the objects in the one with those in the other. Any two classes admitting of such a correlation are said to be *equivalent*. Now given any class K, there is another class C composed of all the classes each of which is equivalent to K. C is called the cardinal number of K, and the name of C, if it have received a name, tells how many objects are in K. Thus, if K is the class of the fingers of my right hand, the word *five* is the name of the class of classes each equivalent to K. Now to the application. The terms of (S) constitute a class K (of terms). Has it a definite number? Yes. What is it? It is the class of all classes each equivalent to K. Has this number-class received a name of its own? Yes, and it has, like many other numbers, received a symbol, namely, $\aleph_0$, read Aleph null. It is, then, this cardinal number Aleph, not familiar, indeed, but perfectly definite as denoting a definite class, it is this that tells us how many terms are in (S) and therewith tells us the length of the rays of our sheaf. Herewith the concept that was forming is now completely formed: *space is a sphere centered at P.*

But is the sphere, as Pascal asserts, an *infinite* sphere? We may easily see that it is. Again consider the sequence (S) and with it the similar sequence (S'),

$$(S)\quad 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \cdots,$$
$$(S')\quad 2,\ 4,\ 6,\ 8, 10, 12, 14,\ \cdots.$$

Observe that all the terms in (S') are in (S) and that (S) contains terms that are not in (S'). (S') is, then, a proper *part* of (S). Next observe that we can pair each term in (S) with the term below it in (S'). That is to say: the whole, (S), is equivalent to one of its parts, (S'). A class that thus has a part to which it is equivalent is said to be infinite, and the number of things in such a class is called an infinite number. Aleph is, then, an infinite number, and so we see that the rays of our sheaf, the radii of our sphere, are infinite in length: *space is an infinite sphere entered at P.*

Finally, what of the phrases, *center everywhere, surface nowhere?* Can we give them a meaning consistent with common usage and common sense? We can, as follows. Let O be any chosen point somewhere in your neighborhood. By saying that the center P is everywhere we mean that P may be taken to be *any* point within a sphere centered at O and having a finite radius, a radius, that is, whose length in miles is expressed by any integer in (S). And by saying that the surface of our infinite sphere is nowhere we mean that no point of the surface can be reached by traveling out from P any *finite* number, however large, of miles, by traveling, that is, a number of miles expressed by any number, however large, in (S).

Here we have touched our primary goal: we have demonstrated that men and women whose education, in respect of space, has been of normal type, believe profoundly, albeit unawares, that the space of our universe is an infinite sphere of which the center is everywhere and the surface nowhere. Such is the beautiful conception,

the great conception—mathematically precise yet mystical withal and awful in its limitless reaches—which is ever ready to form itself, in the normally educated mind and there to stand a deep-rooted conscious conviction regarding the shape and the size of the all-embracing world.

Is the conception valid? Does the conviction correspond to fact? Is it true? It is not enough that it be intelligible, which it is; it is not enough that it be noble and sublime, which also it is. No doubt whatever is noble and sublime is, in some sense, true. For we mortals have to do with more than reason. Yet science, science in the modern technical sense of the term, having elected for its field the domain of the rational, allows no superrational tests of truth to be sufficient or final. We must, therefore, ask: are the dimensions and the figure of our space, in fact, what, as we have seen, Pascal asserts and the normally educated mind believes them to be? Long before the days of Pascal, back yonder in the last century before the beginning of the Christian era, one of the acutest and boldest thinkers of all time, immortal expounder of Epicurean thought, answered the question with the utmost confidence in the affirmative. I refer to Lucretius and his "De Rerum Natura." In my view that poem is the greatest and finest union of literary excellence and scientific spirit to be found in the annals of human thinking. I maintain that opinion of the work despite the fact that the majority of its conclusions have been invalidated by time, have perished by supersession; for we must not forget that, in respect of knowledge, "the present is no more exempt from the sneer of the future than the past has been." I maintain that opinion of the work despite the fact that the enterprise of Lucretius was marvelously extravagant; for we must not forget that the relative modesty

of modern men of science is not inborn, but is only an imperfectly acquired lesson. Well, it is in that great work that Lucretius endeavors to prove that our universe of space is infinite in the sense that we have explained. His argument, which runs to many words, may be briefly paraphrased as follows. Conceive that, starting from any point of space, you go out in any direction as far as you please, and that then you hurl your javelin. Either it will go on, in which case there is space ahead for it to move in, or it will not go on, in which case there must be space ahead to contain whatever prevents its going. In either case, then, however far you may have gone, there is yet space beyond. And so, he concludes, space is infinite, and he triumphantly adds:

Therefore the nature of room and the space of the unfathomable void are such as bright thunderbolts can not race through in their course though gliding on through endless tract of time, no nor lessen one jot the journey that remains to go by all their travel—so huge a room is spread out on all sides for things without any bounds in all directions round.

Such is the argument, the great argument, of the Roman poet. Great I call it, for it is great enough to have fooled all philosophers and men of science for two thousand years. Indeed only a decade ago I heard the argument confidently employed by an American thinker of more than national reputation. But is the argument really fallacious? It is. The conclusion may indeed be quite correct—space may indeed be infinite, as Lucretius asserts—but it does not follow from his argument. To show the fallacy is no difficult feat. Consider a sphere of finite radius. We may suppose it to be very small or intermediate or very large—no matter what its size so long as its radius is finite. By sphere, in this part of the discussion, I shall mean sphere-surface. Be good enough

to note and bear that in mind. Observe that this sphere—this surface—is a kind of room. It is a kind of space, region or room where certain things, as points, circle arcs and countless other configurations can be and move. These things, confined to this surface, which is their world, their universe of space, if you please, enjoy a certain amount, an immense amount, of freedom: the points of this world can move in it hither, thither and yonder; they can move very far, millions and millions of miles, even in the same direction, if only the sphere be taken large enough. I see no reason why we should not, for the sake of vividness, fancy that spherical world inhabited by two-dimensional intelligences conformed to their locus and home just as we are conformed to our own space of three dimensions. I see no reason why we should not fancy those creatures, in the course of their history, to have had their own Democritus and Epicurus, to have had their own Roman republic or empire and in it to have produced the brilliant analogues of our own Vergil, Cicero and Lucretius. Do but note attentively—for this is the point—that their Lucretius could have said about their space precisely what our own said about ours. Their Lucretius could have said to his fellow-inhabitants of the sphere: "starting at any point, go as far as ever you please in any straight line"—such line would of course (as *we* know) be a great circle of the sphere—"and then hurl your javelin"— the javelin would, as *we* know, be only an *arc* of a great circle—"either it will go on, in which case, etc.; or it will not, etc."; thus giving an argument identical with that of our own Lucretius. But what could it avail? *We* know what would happen to the javelin when hurled as supposed in the surface: it would go on for a while, there being nothing to prevent it. But

whether it went on or not, it could not be logically inferred that the surface, the space in question, is infinite, for we know that the surface is finite, just so many, a finite number of, square miles. The fallacy, at length, is bare. It consists—the fact has been recently often pointed out— in the age-long failure to distinguish adequately between *unbegrenzt* and *unendlich* —between *boundless* and *infinite* as applied to space. What our fancied Lucretius proved is, if anything, that the sphere is boundless, but not that it is infinite. What our real Lucretius proved is, if anything, that the space of our universe is boundless, but not that it is infinite. That a region or room may be boundless without being infinite is clearly shown by the sphere (surface). How evident, once it is drawn, the distinction is. And yet it was never drawn, in thinking about the dimensions of space, until in 1854 it was drawn by Riemann in his epoch-marking and epoch-making *Habilitationschrift* on the foundations of geometry.

What, then, is the fact? Is space finite, as Riemann held it may be? Or is it infinite, as Lucretius and Pascal deliberately asserted, and as the normally educated mind, however unconsciously, yet firmly believes? No one knows. The question is one of the few great outstanding scientific questions that intelligent laymen may, with a little expert assistance, contrive to grasp. Shall we ever find the answer? Time is long, and neither science nor philosophy feels constrained to haul down the flag and confess an *ignorabimus*. Neither is it necessary or wise for science and philosophy to camp indefinitely before a problem that they are evidently not yet equipped to solve. They may proceed to related problems, always reserving the right to return with better instruments and added light.

In the present instance, let us suppose,

for the moment, that Lucretius, Pascal and the normally educated mind are right: let us suppose that space is infinite, as they assert and believe. In that case the bounds of the universe are indeed remote, and yet we may ask: are there not ways to pass in thought the walls of even so vast a world? There are such ways. But where and how? For are we not supposing that the walls to be passed are distant by an amount that is infinite? And how may a boundary that is infinitely removed be reached and overpassed? The answer is that there are many infinites of many orders; that infinites are surpassed by other infinites; that infinites, like the stars, differ in glory. This is not rhetoric, it is naked fact. One of the grand achievements of mathematics in the nineteenth century is to have defined infinitude (as above defined) and to have discovered that infinites rise above infinites, in a genuine hierarchy without a summit. In order to show how we can in thought pass the Lucretian and Pascal walls of our universe, I must ask you to assume as a lemma a mathematical proposition which has indeed been rigorously established and is familiar, but the proof of which we can not tarry here to reproduce. Consider all the real numbers from *zero* to *one* inclusive, or, what is tantamount, consider all the points in a unit segment of a continuous straight line. The familiar proposition that I am asking you to assume is that it is not possible to set up a one-to-one correspondence between the points of that segment and the positive integers (in the sequence above given), but that, if you take away from the segment an infinitude (Aleph) of points matching all the positive integers, there will remain in the segment more points, infinitely more, than you have taken away. That means that the infinitude of points in the segment infi-

nitely surpasses the infinitude of positive integers; surpasses, that is, the infinitude of mile posts on the radius of our infinite (Pascal) sphere. Now conceive a straight line containing as many miles as there are points in the segment. You see at once that in that conception you have overleaped the infinitely distant walls of the Lucretian universe. Overleaped, did I say? Nay, you have passed beyond those borders by a distance infinitely greater than the length of any line contained within them. And thus it appears that, not our imagination, indeed, but our reason may gaze into spatial abysses beside which the infinite space of Lucretius and Pascal is but a meager thing, infinitesimally small. There remain yet other ways by which we are able to escape the infinite confines of this latter space. One of these ways is provided in the conception of hyperspaces enclosing our own as this encloses a plane. But that is another story, and the hour is spent.

The course we have here pursued has not, indeed, enabled us to answer with final assurance the two questions with which we set out. I hope we have seen along the way something of the possibilities involved. I hope we have gained some insight into the meaning of the questions and have seen that it is possible to discuss them profitably. And especially I hope that we have seen afresh, what we have always to be learning again, that it is not in the world of sense, however precious it is and ineffably wonderful and beautiful, nor yet in the still finer and ampler world of imagination, but it is in the world of conception and thought that the human intellect attains its appropriate freedom—a freedom without any limitation save the necessity of being consistent. Consistency, however, is only a prosaic name for a limitation which, in

another and higher realm, harmony imposes even upon the muses.

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CLINICAL PSYCHOLOGY: WHAT IT IS AND WHAT IT IS NOT

ON an occasion like this[1] it would seem proper, representing as I do one of the newest of the sciences, that I address myself to some of the basic questions of this science. Perhaps the very first question with which one is confronted is simply this: "In view of the rapid multiplication of the sciences, by what right does clinical psychology lay claim to an independent existence?" That is a question which may perturb some sensitive minds, but it does not disconcert the clinical psychologist, for he regards the question as perfectly legitimate and capable of satisfactory answer.

It is just and proper that a new claimant to membership in the family of sciences should be required to present her creden-

[1] Substance of an address delivered before the Conference on the Exceptional Child, held under the auspices of the University of Pittsburgh, April 16, 1912. Lest misapprehensions arise, it should be clearly understood that in this discussion I am concerned only with the relation of *clinical psychology* to *mentally* exceptional school children; and that I distinctly recognize a different type of exceptional children, namely, the *physical* defectives. The physical defectives should be examined by skilled pediatricians. The clinical psychologist is interested in physically exceptional children if they manifest mental deviations. Moreover, while I hold that the psycho-clinical laboratories must become the clearing houses for all types of *mentally* or *educationally* exceptional children in the schools, nearly all mentally exceptional children should be given a physical examination by consulting or associated medical experts. Physical abnormalities should, of course, be rectified, whether or not it can be shown that they sustain any causal relation to any mental deviations which may have been disclosed in the psycho-clinical examination. They may claim treatment in their own right.

tials. It is a natural human trait to challenge or contest the claims of a newcomer. It has ever been thus. Every branch of knowledge before winning recognition as an independent science has been forced to demonstrate that it possesses *a distinct and unique body of facts* not adequately treated by any other existing science; or that it approaches the study of a *common body* of facts from a *unique* point of view, and with methods of its own. Psychology, biochemistry, dentistry, eugenics, historiometry and many other sciences have been thus obliged to fight their way inch by inch to recognition as independent sciences. It is not long since physiology claimed psychology as its own child and stoutly contested her rights to existence; nor is it long since medicine denied any right to independent existence to dentistry. It is no surprise that a number of sciences now claim clinical psychology as part and parcel of their own flesh and blood, and that they deny her the right to "split off from the parent cell" and establish an unnursed existence of her own. Just as nature abhors a vacuum, so science abhors the multiplication of sciences; just as the big corporation octopus in the industrial world tries to get monopolistic control of the sources of production and distribution, so the various sciences, naturally insatiable in their desire for conquest, attempt only too often to get monopolistic control of all those elements of knowledge which they may be able to use for their own aggrandizement, whether or not they have developed adequate instruments for scientifically handling those elements.

Clinical psychology, however, is quite ready to contest the attempts to deprive her of her inalienable rights to the "pursuit of life and happiness." Fundamentally, she bases her claims to recognition as an independent science on the fact that